Bode Plots

This note will cover the basics of plotting transfer functions using Bode plots. We will use these to graph the response of filters at different frequencies. For a review on filters and the transfer functions that represent them, see the note on filters posted last week.

Plotting Basics

Bode plots consist of two parts: the magnitude response and the phase response. These are plotted on their own graphs, each with a different scale. Both plots have ω on the horizontal axis, as this is the input of the transfer function. We plot the frequency on a logarithmic scale to fully show the range of frequencies that the filer essentially captures. For the phase plot, the vertical axis can be either in degrees or in radians.

For the magnitude plot, we choose to represent the magnitude of the transfer function in decibels. Specifically, the transfer function is as follows:

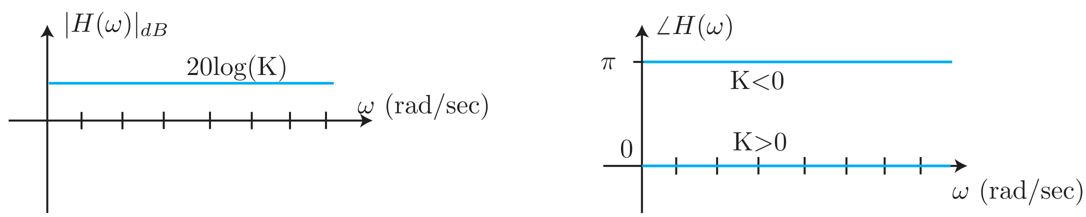
$$|H(\omega)|_{dB} = 20\log|H(\omega)| \tag{1}$$

You may be wondering why we use a factor of 20 instead of 10 (since the unit is a "deci"bel). This is by definition: decibels are defined from gains in power, while the transfer function expresses voltage gain. Therefore, because power is proportional to the square of the voltage or the current, we get an extra factor of 2 in front, leading to the 20 factor as opposed to 10.

Types of plots

There are five general types of plots:

1. **Constants**



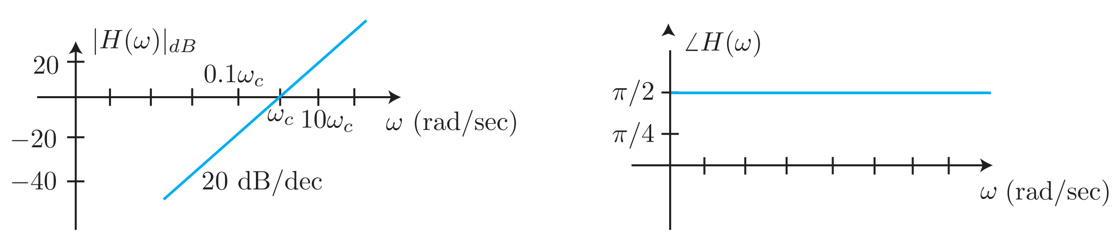
Constant terms are the easiest to plot. Because you only have a real value, the magnitude of the transfer function is equal to

$$|H(\omega)|_{dB} = 20\log|K| \tag{2}$$

Because there is no dependency to frequency, the output is linear across the plot.

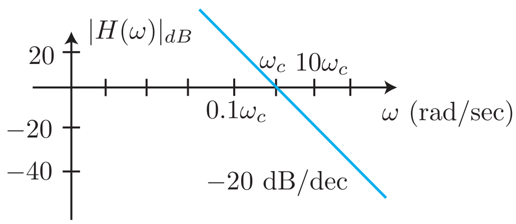
Since a real constant has no imaginary part, we note that it lies on the real axis on the complex plane. Therefore, there will be a phase of 0 for positive constants and π for negative ones.

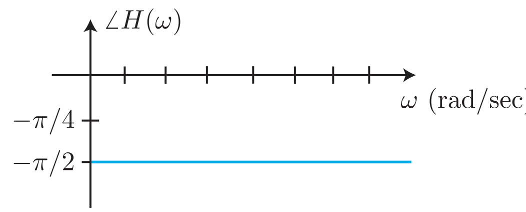
2. **Zeros**



Zeros take the form $j\omega/\omega_c$. We will note first the intercept where $\omega = \omega_c$. The cutoff frequency is where the magnitude of the transfer function will equal 1. Taking the log of this results in the intercept where $|H(\omega_c)|_{dB} = 0$. We also see that the function linearly increases with respect to the frequency, so a 10-fold increase in frequency results in a 20 dB increase in the magnitude of the transfer function. In terms of phase, we note that our only term is the imaginary j term. Therefore, the phase response will be $\pi/2$ for all frequencies.

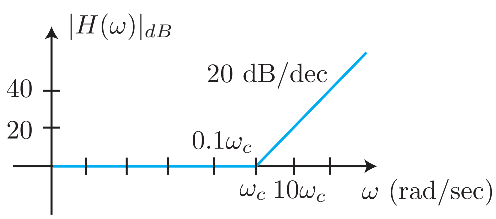
3. **Poles**

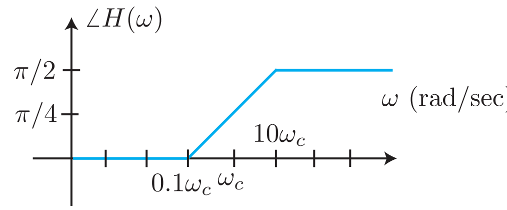




Poles are of the form $\frac{1}{j\omega/\omega_c}$. We note that the pole is merely the inverse of the zero, so a 10 fold increase in frequency results in a 20 dB *decrease* in the magnitude. Therefore, one can reflect the graph along the x-axis and note your output has the same value at the cutoff frequency, but decreases at 20 dB/dec. With the phase, we note that because the j term is in the denominator, the vector is going in the negative j direction, so the phase will be $-\pi/2$.

4. **Simple Zeros**



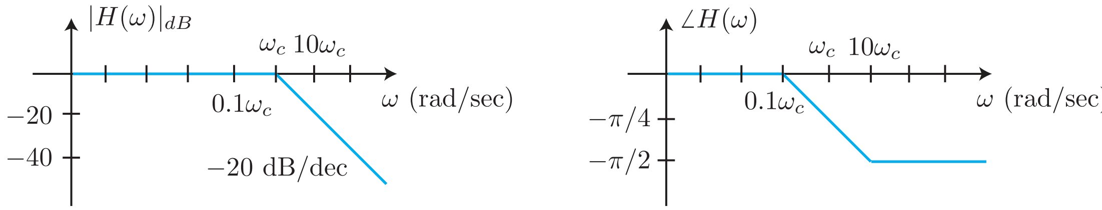


The transfer function of the simple zero takes the form $1 + j\omega/\omega_c$. For low frequencies, the response will be dominated by the real term compared to the imaginary term, so it will look like a constant. For higher frequencies, the imaginary frequency term will be dominant against the real term, so it will look like a zero.

At the cutoff frequency $\omega = \omega_c$, we see that the transfer function goes to $1 + j$. This corresponds to a magnitude of $\sqrt(2)$ and a phase of $\pi/4$. Note that in reality, the magnitude will be at 3 dB instead of 0 dB, so some error does exist, but for graphing purposes, we approximate that they about the same.

In terms of the phase shifts, we note again what the values are at the extremes. At low frequencies (as ω approaches 0), the real term dominates, so there is no phase shift, but as the frequency approaches infinity, the imaginary term dominates, so the phase goes to $\pi/2$. We typically consider the phase to reach 0 around 1/10 of the cutoff frequency, and $pi/2$ around 10 times the cutoff frequency. Like the phase value at the cutoff frequency, these linear connections are approximations, but they can serve as a good estimate in the phase response.

5. **Simple Poles**



Just as the pole is the inverse of the zero, the simple pole is the inverse of the simple zero. The transfer function is given in the form $\frac{1}{1+j\omega/\omega_c}$. We can apply the same ideas seen in simple zeros by analyzing the form of the transfer function as it hits the extreme frequency values.

For high frequencies, the magnitude response will decrease at 20 dB per decade, while the corresponding phase shifts down to $-\pi/2$. Like the simple zero, the cutoff frequency has a value of -3 dB that can be approximated as the intersection of the constant and the pole components. The phase at the cutoff frequency is $-\pi/4$. It goes to 0 at $0.1\omega_c$ and $-\pi/2$ at $10\omega_c$.

Superposition

One useful math trick we can use is the superposition of Bode plots. We know that the log of a multiplied term is equivalent to the sum of the logs of the terms, as seen in this form: $\log(cd) = \log(c) + \log(d)$. Because we can superimpose the logs of the multiplied terms, we will typically try to get transfer function in terms of poles and zeros. From there we can plot each individual function and sum up the resultant graphs to get your final frequency response.

Summary

You have seen various kinds of filters: the LPF, HPF, BPF, etc. These transfer functions may seem complex and their corresponding plots nontrivial, but you may be able to break up the transfer functions into poles and zeros. If that is the case, then you can plot each individual pole and zero with their respective cutoff frequencies and apply superposition. In this way, you should be able to plot many higher-order filters as well!

In reading Bode plots, you should be able to note what frequencies you have for your input signal. Then you can just read off the graphs and find the magnitude and frequency response, saving a fair amount of time as opposed to finding the magnitude and phase of every frequency of the input source.