Filters

Hopefully by this point, you have a pretty good idea of what filters are and what they do. In these notes we will review the different types of filters and their characteristics, as well as some of their applications. A very important and related concept is the graphical representation of filters using *Bode plots*. Details on how to construct and use them are covered in separate notes.

Recall that any circuit that has an input and output defined can basically be called a filter. Filters can be characterized by their *transfer function*:

$$H(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_{in}} \tag{1}$$

It is important to note that H is complex function of ω. This definition looks similar to the gain of an amplifier that we saw often back with resistive circuits, but the gains that we saw before were real and constants. Here we are working with phasors (hence complex), and our circuit behaviors depend on complex impedances (hence the dependence on ω). Because of linearity, we can easily find the output of a filter in the time domain. Given an input $v_{in}(t) = V_m \cos(\omega t + \theta_v)$, the output is simply

$$v_{out}(t) = |H(\omega)| \times V_m \cos(\omega t + \theta_v + \angle H(\omega)) \tag{2}$$

This should be apparent, since $\mathbf{V}_o$ in the phasor domain is simply the product of H and $\mathbf{V}_{in}$. Hence we can multiply their magnitudes and add their phases. Again, notice that the new response is a function of the frequency ω; for a function with a different frequency, both the magnitude and phase can be different. The other observation you should make is that the frequency does not change from input to output. Hence we can always apply this "shortcut" to find the output.

First-Order Filters

The simplest filters can be implemented with RC and RL circuits. The key to analyzing them by inspection is to remember the behavior of inductors and capacitors. Remember that inductors pass low frequencies and block high frequencies, while capacitors do the opposite. Thus, the following circuits implement *lowpass filters*, passing low-frequency signals and killing high-frequency ones.

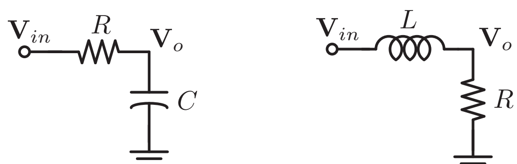

Figure 1: The RC and RL lowpass filters

One can easily derive the transfer functions for the above two filters. If we define the *cutoff frequency* ω_c for each circuit such that $\omega_c = \frac{1}{RC}$ for the first case and $\omega_c = \frac{R}{L}$, then both have a transfer function of

$$H(\omega) = \frac{1}{1 + j\omega/\omega_c} \tag{3}$$

The Bode magnitude and phase plots for the transfer function are shown below. Notice that the magnitude plot confirms what we already know—low frequencies below ω_c are largely unaffected, while the magnitude decreases at $-20\,\text{dB/dec}$ for frequencies above ω_c. A signal would also have its phase shifted for high frequencies, with the most extreme shift occurring at frequencies of $10\omega_c$ or higher.

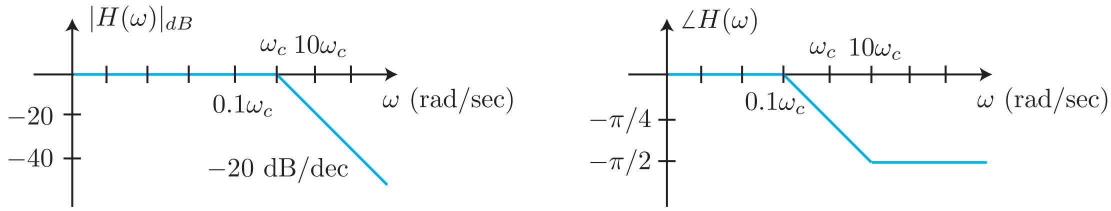

Figure 2: The magnitude and phase plots for the LPF

On the other hand, the *highpass filter* equivalents of the above two circuits are the following:

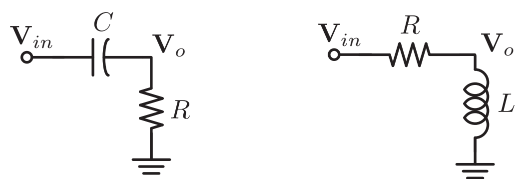

Figure 3: The RC and RL highpass filters

Again defining the same cutoff frequencies as before, it can be shown that the transfer function is

$$H(\omega) = \frac{j\omega/\omega_c}{1 + j\omega/\omega_c} \tag{4}$$

The Bode plots are shown below. Notice that the effect is opposite to that of the lowpass filter. High frequencies are largely untouched, in both magnitude and phase, while low frequencies are increasingly attenuated and phase-shifted.

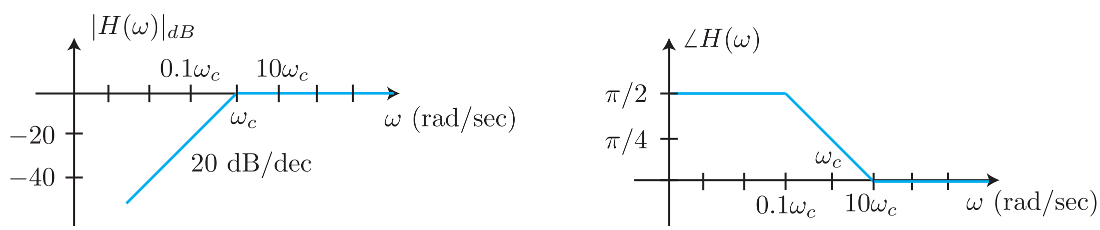

Figure 4: The magnitude and phase plots for the HPF

Although the above circuits may seem pretty trivial, many variations of the LPF and HPF can be constructed by combining multiple elements together. Any filter whose transfer function contains at most one pole and one zero can be classified as a *first-order filter*. Such filters usually only implement either the LPF or HPF, with a magnitude rolloff of 20 dB/dec. It is not possible to get a steeper rolloff (e.g. 40 dB/dec) because each zero or pole alone only contributes 20 dB/dec.

Higher-Order Filters

By combining multiple capacitors and/or inductors together in one filter circuit, one can construct filters of much higher order and obtain a steeper rolloff than ± 20 dB/dec. These higher-order filters can be useful if there is a greater need to kill off unwanted frequencies more quickly. Sometimes a rolloff of ± 20 dB/dec may not be enough if we wished to remove frequencies close to our cutoff frequency. To illustrate this, consider the circuit below:

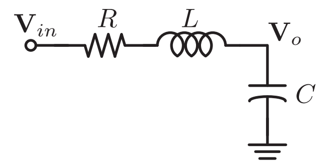

Figure 5: Second-order lowpass filter

As an exercise, show that the transfer function can be expressed as

$$H(\omega) = \frac{1}{1 + j\omega RC - \omega^2 LC} \tag{5}$$

Notice from both the circuit and the transfer function that we still have a LPF; as $\omega \to 0$, $H(\omega) \to 1$, while $H(\omega) \to 0$ as ω gets large. The difference from before is that we now have *two* poles as opposed to one before. Hence, at high frequencies, when the ω^2 term dominates the denominator, we can see that increasing ω by a factor of 10 leads to a decrease in $|H(\omega)|$ by a factor of 100. This translates to 40 in decibals, giving us a rolloff of $-40\,\text{dB/dec}$.

Think about this once more as you consider the following second-order HPF:

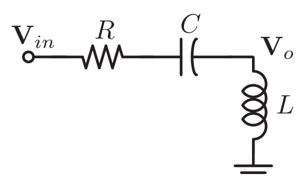

Figure 6: Second-order highpass filter

The transfer function can be shown to be

$$H(\omega) = \frac{-\omega^2 LC}{1 + j\omega RC - \omega^2 LC} \tag{6}$$

As $\omega \to 0$, $H(\omega)$ clearly also goes to 0 due to the numerator. But as ω becomes large, the magnitude of $H(\omega)$ becomes closer and closer to 1 as the lower-order terms in the denominator become negligible. For small ω, however, notice that the ω^2 term would naturally lead to a slope of $40\,\text{dB/dec}$ just as before.

We can also now design two new types of filters using second-order circuits. Consider the following *bandpass filter*:

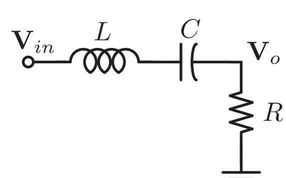

Figure 7: Series RLC bandpass filter

For both small and large ω, the magnitude of this function goes to 0 (alternatively, the capacitor will block low frequencies, while the inductor will block high ones). Its transfer function is given by

$$H(\omega) = \frac{j\omega RC}{1 + j\omega RC - \omega^2 LC} \tag{7}$$

Thus we see that the magnitude of $H(\omega)$ goes to 0 for both frequency extremes, particularly as $\omega \to \infty$ since the denominator is second-order.

However, this transfer function's magnitude is close to 1 for a small range of frequencies around the *resonant frequency* $\omega_0 = \frac{1}{\sqrt{LC}}$:

$$H(\omega_0) = \frac{j\omega RC}{1 + j\omega RC - (1/\sqrt{LC})^2 LC} = \frac{j\omega RC}{1 + j\omega RC - 1} = \frac{j\omega RC}{j\omega RC} = 1 \quad (8)$$

This is also the frequency for which the impedances of the capacitor and inductor cancel, leaving only the resistance. Thus maximal current can flow, leading to the maximum voltage across the resistance. This circuit only allows a small band of frequencies near its resonant frequency to pass through, hence leading to its name.

The opposite of a bandpass filter is a *bandreject* or *bandstop filter*:

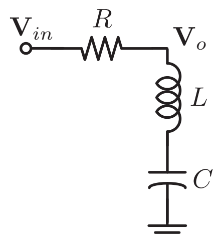

Figure 8: Series RLC bandstop filter

As before, its transfer function is

$$H(\omega) = \frac{1 - \omega^2 LC}{1 + j\omega RC - \omega^2 LC} \quad (9)$$

So it passes frequencies at both extremes (since either the inductor or the capacitor will shield the output from ground). However, around the resonant frequency, once again $\omega_0 = \frac{1}{\sqrt{LC}}$, we see that the magnitude actually goes to 0:

$$H(\omega_0) = \frac{1 - (1/\sqrt{LC})^2 LC}{1 + j\omega RC - (\sqrt{LC})^2 LC} = \frac{1 - 1}{1 + j\omega RC - 1} = 0 \quad (10)$$

Thus, we say that this filter, also called a notch filter, rejects or stops this band of frequencies from passing through. Like bandpass filters, they are useful if we only care about a specific range of frequencies, but this time we are killing them rather than passing them.

A common practice is to *cascade* two or more filters together to filter a signal more than once. For example, a BPF can be achieved by first passing the input through a LPF to get rid of the low frequencies, followed by a HPF to kill the high ones. The following circuit shows this example using resistors and capacitors only:

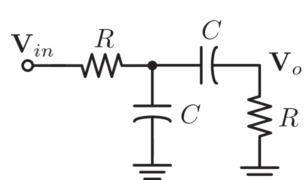

Figure 9: BPF formed using a cascade of RC circuits

Note that one can pick different values for the components to achieve different cutoff frequencies. Choosing the right cutoff frequencies is often a key step in filter design. For the example above, $\omega_c = \frac{1}{RC}$ for both parts, but one can easily shift the two cutoffs away from each other to allow a higher passband.

Active Filters

All of our filters so far have a maximal magnitude of 1. This is because they are all *passive*; without any active elements or sources, there is no way for the output to be greater than the input. But we can change this by throwing in elements such as op amps. Consider the following circuit:

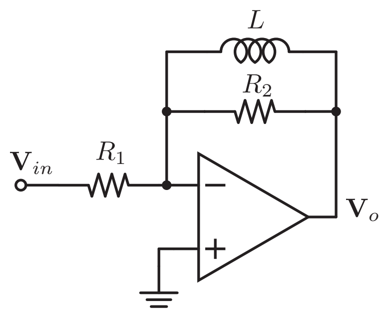

Figure 10: First-order active HPF

Applying the summing point constraint at $\mathbf{V}_-$ just like with resistive amplifiers before, we can find the following transfer function:

$$H(\omega) = -\frac{R_2}{R_1}\left(\frac{j\omega/\omega_c}{1 + j\omega/\omega_c}\right) \tag{11}$$

where $\omega_c = \frac{R}{L}$ as before. Notice that the constant aside, this is just the transfer function for a HPF. Indeed, as $\omega \to 0$, the inductor shorts out, linking $\mathbf{V}_o$ to $\mathbf{V}_- = 0$. But now, for $\omega \to \infty$, we actually have $H(\omega) \to -\frac{R_2}{R_1}$, which can potentially be greater than 1! Of course, this should make sense, as this is just an inverting amplifier if one considers the inductor to be an open circuit. This active filter thus both filters and gains input signals.

With active filters, one can easily construct more interesting examples of higher-order filters and cascades of filters. Consider the following example:

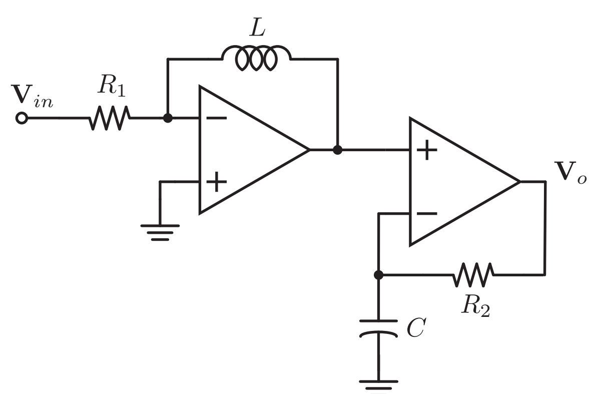

Figure 11: Second-order active cascade HPF

While apparently intimidating, look more carefully and recognize that this is just simply a cascade of an inverting and noninverting op amp configuration. Either using KCL or remembering the gains of these amplifiers, we can derive the transfer function:

$$H(\omega) = \left(-\frac{j\omega L}{R}\right)(1 + j\omega RC) = -j\omega\frac{L}{R} + \omega^2 LC \tag{12}$$

Clearly, this filter is second-order due to the ω^2 term. Its magnitude also increases without bound as $\omega \to \infty$, making this a highpass filter.

Summary and Applications

We've barely scratched the surface of the theory and applications of filters, but hopefully you now have some sense of their operation. In terms of theory, there are many common filter configurations that are often used. Most of them are at least second-order. Some common ones include the following:

- Chebyshev filter
- Butterworth filter
- Bessel filter
- Elliptic filter

We've also barely analyzed the various qualities of filters. All the ones we've seen so far are analog and LTI (linear, time-invariant), but one can also construct digital filters, nonlinear filters, or time-variant filters. One can also talk about the bandwidth or Q-factor (quality factor) of higher-order filters, which describe how well they do their job in passing desired frequencies and killing others. We also have not discussed the implications of phase shifting in terms of the phase margin, which has important implications for the *stability* of the system of interest.

Mathematics aside (see EE120 if you are interested), filters arguably are important for a number of different real-world applications. For example, audio filters are used for graphic equalizers, synthesizers, sound effects, etc. Without filters to clean up all the music that you listen to, everything will sound like crap.

Line filters are often used in powering electronic equipment. Electromagnetic interference (EMI) is often a problem that arises when transmitting AC power across a line to a load. Using filtration techniques, one can get rid of these unwanted frequencies before they make it to the load.

Image processing relies heavily on filters to do the job. By filtering out high frequencies, an image of pixels can be made sharper by focusing on visual edges and minimizing unwanted qualities such as blurriness, shimmering, and blocking. Such texture filtering helps make images much more visually appealing.

When using sensors and devices to measure signals, one will inevitably pick up high-frequency noise, whether from the environment or from the measuring equipment. Since this unwanted noise corrupts the input that we want, it is important to be able to get rid of it using filtering techniques without touching the original signal.

Hopefully by now we've managed to convince you of the importance of filters. Don't be surprised if you find yourself having to apply this knowledge some day, whether electronically or mathematically, as their applications pop up literally everywhere.