First-Order RC and RL Transient Circuits

When we studied resistive circuits, we never really explored the concept of *transients*, or circuit responses to sudden changes in a circuit. That is not to say we couldn't have done so; rather, it was not very interesting, as purely resistive circuits have no concept of time. That is, if we consider an arbitrary switch action in a resistive circuit, we would simply apply our circuit analysis techniques to the circuit before and after the switch action. The new values will likely be different from the old ones, but there is no notion of *how* we got to the new values from the old ones. We simply assume an instantaneous, discontinuous jump.

Not so with circuits containing capacitors and inductors. These devices have *i*-*v* characteristics which contain explicit dependencies on time. The voltage or current of one of these devices depends not only on some other quantity at this moment, but also on a quantity *in the past*. Such dependencies are captured through the derivative and integral operators. Hence we cannot have instantaneous changes in certain quantities.

$$i_C(t) = C\frac{dv_C(t)}{dt} \tag{1}$$

$$v_L(t) = L\frac{di_L(t)}{dt} \tag{2}$$

The Canonical Charging and Discharging RC Circuits

Consider two different circuits containing both a resistor R and a capacitor C. One circuit also contains a constant voltage source V_s; here, the capacitor C is initially uncharged. In the other circuit, there is no voltage source and the capacitor is initially charged to V_0.

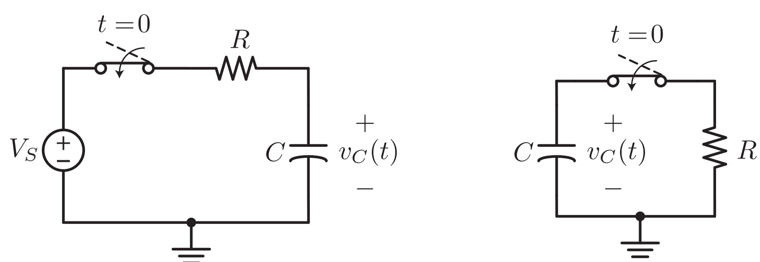

Figure 1: The charging and discharging RC circuits

In both cases, the switch has been open for a long time, and then we flip it at time $t = 0$. What happens in the circuit throughout the entire experiment? In particular, let's focus on $v_C(t)$, as knowing that will also give us the current $i_C(t)$ by equation 1 above. If we follow the same methodology as with resistive circuits, then we'd solve for $v_C(t)$ both before and after the switch closes.

Well, before the switch closes, both circuits are in an open state. So $v_C(0_-)$ for the uncharged capacitor is just 0, while it is V_0 for the charged capacitor.

After the switch closes, we have complete circuits in both cases. KCL at the node v_C gives us the two equations for the charging and discharging circuits, respectively:

$$v_C(t) + RC\frac{dv_C(t)}{dt} = V_s \tag{3}$$

$$v_C(t) + RC\frac{dv_C(t)}{dt} = 0 \tag{4}$$

Notice that we cannot simply solve an algebraic equation and end up with a single value for v_C anymore. Instead, $v_C(t)$ is given by an ordinary differential equation that depends on time. Hence, the function $v_C(t)$ describes the *transient* response after the switch closes; it is not instantaneous. The other observation you should make is that the equations for both cases are strikingly similar. The task that is now left to us is to solve these ODEs.

Solving First-Order Ordinary Differential Equations

The general form of the first-order ODE that we are interested in is the following:

$$x(t) + \tau \frac{dx(t)}{dt} = f(t) \tag{5}$$

Here, the *time constant* τ and the *forcing function* $f(t)$ are given, and we are solving for $x(t)$. ODE theory tells us that there are two separate solutions to the above equation, and the general solution is the superposition of the two. First we consider the *homogeneous equation*:

$$x(t) + \tau \frac{dx(t)}{dt} = 0 \tag{6}$$

Notice that the only difference from the original equation 5 is that the RHS is 0. The solution to this can be found by substitution or direct integration. This is known as the *complementary solution*, or the *natural response* of the circuit in the absence of any active sources:

$$x_c(t) = Ke^{-t/\tau} \tag{7}$$

Clearly, the natural response of a circuit is to decay to 0. Hence, without any sources present, any capacitor (inductor) will eventually discharge until it has no voltage (current) left across it.

The other solution of the ODE is the *particular solution* or *forced response* $x_p(t)$, due to the forcing function. Unlike the complementary solution, we have no general formula for finding this. The only headway we have is that $x_p(t)$ takes the same form as that of $f(t)$. This must hold as $x_p(t)$ appears on the LHS of the ODE, along with its derivative, and their linear combination must equal $f(t)$. Thus, one would assume a solution $x_p(t)$ of the form of $f(t)$ plus its derivative.

Some examples of particular solutions are shown below. Notice that we always assign arbitrary constants to each term to preserve generality.

- If $f(t)$ is constant, e.g. $f(t) = 1$, assume $x_p(t) = A$
- If $f(t)$ is linear, e.g. $f(t) = 3t$, assume $x_p(t) = At + B$
- If $f(t)$ is quadratic, e.g. $f(t) = -2t^2 + 5t$, assume $x_p(t) = At^2 + Bt + C$
- If $f(t)$ is exponential, e.g. $f(t) = 5e^{-\omega t}$, assume $x_p(t) = Ae^{-\omega t}$
- If $f(t)$ is sinusoidal, e.g. $f(t) = 7\cos\omega t$, assume $x_p(t) = A\sin\omega t + B\cos\omega t$

The list goes on and on. The majority of the circuits you will see here, however, only involve DC sources, which means $f(t)$ will almost always be constant. The final step is to add both the complementary and particular solutions together for the complete solution to the original ODE.

$$x(t) = x_c(t) + x_p(t) \tag{8}$$

Because we know that $x_c(t)$ worked due to it evaluating the LHS of the ODE to 0, we can add this to $x_p(t)$, and the subsequent function $x(t)$ should stil satisfy the ODE. What we gained here was an even more general solution for the ODE. Finally, we still have a whole bunch of constants floating around. These we can solve for by using the initial and final conditions of the circuit and plugging them into the function $x(t)$.

Note that you will not always have the opportunity to use final conditions, as you do not always necessarily know what they are. For DC sources, it is correct to apply DC steady-state analysis if you are simply solving for $x(\infty)$. However, we would not be able to infer the final condition of a circuit with a non-DC source. In addition, when using initial conditions you must take care that your assumptions make physical sense. For example, voltage cannot change instantaneously across a capacitor, but no such restriction exists for the current.

An alternative way to solve for the constants is to plug the solution $x(t)$ back into the original ODE. This would yield another equation to use in solving for the constants.

Back to the RC Circuits

Using our handy guide above, we conclude that the solution (both complementary and particular) to the ODEs 3 and 4 looks like this:

$$v_C(t) = Ke^{-t/RC} + A \tag{9}$$

The charging case gives us boundary conditions $v_C(0) = 0$, as we know the voltage value immediately before the switch closes, and $v_C(\infty) = V_s$, as the capacitor becomes an open circuit at DC steady-state and causes all of the source voltage to appear across it. Using these two conditions, we obtain the solution for the charging capacitor:

$$v_C(t) = V_s(1 - e^{-t/RC}) \tag{10}$$

On the other hand, the discharging capacitor has boundary conditions $v_C(0) = V_0$ and $v_C(\infty) = 0$, since we expect the capacitor to have completely discharged after a long time. Plugging these in and solving for constants thus gives us the discharging solution:

$$v_C(t) = V_0 e^{-t/RC} \tag{11}$$

We thus conclude that the first-order transient behavior of RC (and RL, as we'll see) circuits is governed by decaying exponential functions. Instead of changing immediately, it takes some time for the charge on a capacitor to move onto or off the plates. This exponential behavior can also be explained physically.

In the charging case, the voltage initially rises quickly because there is little charge on the capacitor to oppose the further piling on of charge onto the plate. As the voltage increases, however, it becomes harder for more charges to get onto the plates, hence leading to the exponential slowdown toward the final value, which is solely driven by our voltage source forcing function.

In the discharging case, the voltage initially falls quickly. In the absence of other driving forces, the initial buildup on the capacitor pushes the charge off at a greater rate. But as the voltage decreases, there is less "force" driving the charges off, hence leading to an exponential slowdown.

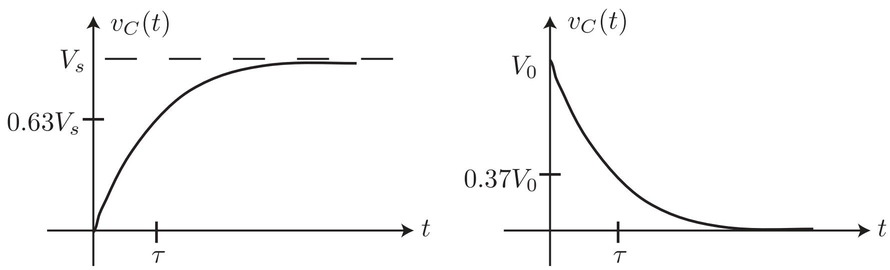

Figure 2: Solutions to the charging and discharging RC circuits

Notice in both cases that the time constant is $\tau = RC$. In other words, how fast or how slow the (dis)charging occurs depends on how large the resistance and capacitance are. One time constant gives us $e^{-\tau/\tau} = e^{-1} \approx 0.37$, which translates to $v_C(\tau) = 0.63V_s$ and $v_C(\tau) = 0.37V_0$ in the charging and discharging cases, respectively. So we say that the circuit is 63% toward its final value after about one time constant. Although these exponentials asymptotically approach these final values and never exactly reach it, we can pretty much approximate that they do so after about three time constants. At that point, we have $v_C(3\tau) = 0.95V_s$ and $v_C(3\tau) = 0.05V_0$ for each of the two cases.

Lastly, what if we had wanted to find another quantity, like the current $i_C(t)$? Since we have the voltage, it is a simple matter to use equation 1 and differentiate the voltage to obtain it. The other option could have been to solve for it initially without solving for voltage, for example by writing a KVL loop instead. For the discharging case, we would get the following equation:

$$Ri_C(t) + \frac{1}{C}\int i_C(t)dt = 0 \tag{12}$$

While this is not an ODE, recall that we can differentiate the equation and rearrange to obtain

$$i_C(t) + RC\frac{di_C(t)}{dt} = 0 \tag{13}$$

which is exactly identical to the equation for the voltage. So we can assume the solution form $i_C(t) = Ke^{-t/\tau}$ and solve for the constant as before. However, note that the initial condition is *not* 0! The current was obviously 0 right before the switch closed, but this tells us nothing about its value immediately afterward. In fact, the initial current is given by Ohm's law across the resistor, since the capacitor's voltage appears across it. Hence the initial current should be $i(0_+) = V_0/R$, which is obviously discontinuous from its previous value.

If we differentiate 11 directly to find $i_C(t)$, we have that the solution should be

$$i_C(t) = \frac{V_0}{R}e^{-t/RC} \tag{14}$$

which agrees with our observation above.

RL Circuits

First-order circuits with inductors can be analyzed in much the same way. Consider the "charging" and "discharging" RL circuits below.

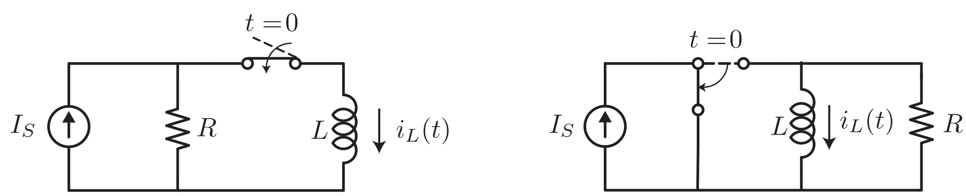

Figure 3: The "charging" and "discharging" RL circuits

While the notion of charging an inductor doesn't really make sense, one can think of this in terms of current. In DC steady-state, inductors act as shorts and allow any current to flow through them, but inductors oppose immediate changes in current and introduce delays between the initial and final currents. Again, these time transient responses are given by decaying exponentials. First note that we can derive KVL equations as follows:

$$i_L(t) + \frac{L}{R}\frac{di_L(t)}{dt} = I_S \tag{15}$$

$$i_L(t) + \frac{L}{R}\frac{di_L(t)}{dt} = 0 \tag{16}$$

Aside from the time constant, these equations are exactly the same as those for the voltage in a RC circuit. Furthermore, the boundary conditions are analogous; in the charging case, $i_L(0) = 0$ and $i_L(\infty) = I_S$, while for discharging we have $i_L(0) = I_S$ and $i_L(\infty) = 0$. The solutions to the ODEs 15 and 16 are just

$$i_L(t) = I_S(1 - e^{-t/\tau}) \tag{17}$$

$$i_L(t) = I_S e^{-t/\tau} \tag{18}$$

All of the analyses that we applied to RC circuits can be applied to RL circuits as well, the only differences being that we are dealing with current and that the time constant is $\tau = \frac{L}{R}$. Physically, the inductor in the first circuit originally had no current flowing through it, but it must eventually end up as a short-circuit in DC steady-state. So it takes some time for the device to "charge up" and allow current to increase to its final value. In the second circuit, the inductor originally had I_S flowing through it, since is a short after being in DC steady-state for a long time. Even though it becomes disconnected from the current source, it tries to keep some current going for a while until it decays to 0.

Again, it is important to note that only current is restricted to not changing instantaneously; no such condition holds for the voltage across an inductor. As with a capacitor, this also turns out to be discontinuous; it is an exercise left to the reader to show that the voltage $v_L(t)$ in each case is actually given by

$$v_L(t) = RI_S e^{-t/\tau} \tag{19}$$

$$v_L(t) = -RI_S e^{-t/\tau} \tag{20}$$

Circuits with Multiple Resistors, Sources, and Switches

While the examples that we analyzed were simple and cute, RC and RL circuits can quickly get ugly, as with resistive and amplifier circuits. However, the same techniques that we've used here can be extended to any first-order circuit.

Look back at the RC and RL circuit diagrams. The ODEs that we obtained only apply to these circuit configurations, but do they look familiar? Indeed, the capacitor sees a Thévenin circuit, while the inductor sees a Norton circuit! Since we know that any linear circuit has these equivalent circuits, this gives an alternative method for writing down the ODE. Given any circuit with a capacitor or inductor, we can reduce the rest of the circuit down to a Thévenin or Norton equivalent. Thus, for *any* arbitrary RC or RL circuit with a single capacitor or inductor, the governing ODEs are

$$v_C(t) + R_{Th}C\frac{dv_C(t)}{dt} = v_{Th}(t) \tag{21}$$

$$i_L(t) + \frac{L}{R_N}\frac{di_L(t)}{dt} = i_N(t) \tag{22}$$

where the Thévenin and Norton circuits are those *as seen by* the capacitor or inductor.

Before getting too excited, do note that this technique is rarely employed to its fullest extent, since we never actually require the entire governing ODE. The only pieces that we need from it are the time constant and the *form* of the forcing function. We can usually guess the latter by simple inspection of the circuit sources. In addition, it is often easier to just use analysis techniques like KCL or KVL to write ODEs for $v_C(t)$ and $i_L(t)$ directly, from which we can also find the time constant.

The greatest advantage that this technique provides us is the ability to find the time constant without writing down any ODEs, which then gives us the complementary solution. This is especially useful when the circuit contains no dependent sources. Recall that in this case $R_{Th} = R_{eq}$ with all sources zeroed out. This is often the quicker way to solve for the time constant, as we would just find R_{eq} as seen by the capacitor or inductor.

Consider the following example, which exemplifies the title of this section.

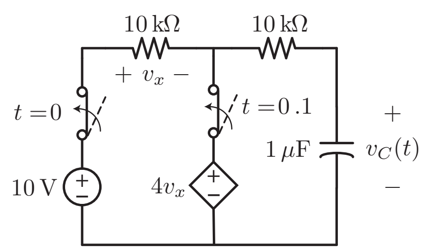

Figure 4: RC Circuit Example

Clearly, since the two switches close at different times, the capacitor will have two distinct behaviors, one before and one after $t = 0.1$ s. Before $t = 0.1$ s, the capacitor has no idea that the second switch exists. Hence we obtain the solution as if the middle branch were not there.

First we obtain the time constant, either from circuit analysis or from finding the equivalent resistance as seen by the capacitor (with the voltage source zeroed out). This gives us $\tau = 0.02$ s. Since the source is DC, the solution takes the form $v_C(t) = Ke^{-t/\tau} + A$. Now we assume the capacitor is initially uncharged, so $v_C(0) = 0$. The final condition, again because the capacitor has no notion that the second switch exists, is $v_C(\infty) = 10$. The solution is thus $v_C(t) = 10(1 - e^{-50t})$ V.

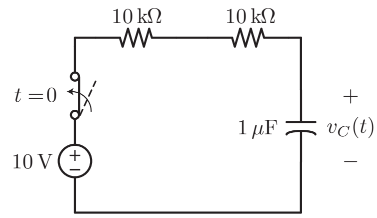

Figure 5: Circuit during the time period $0 \leq t \leq 0.1\,\text{s}$

Now the second switch closes at $t = 0.1\,\text{s}$. Because the circuit has changed, the time constant may also have as well. The presence of the dependent source complicates our solving for R_{Th}, but it can still be done (or we can find an ODE for $v_C(t)$), and we have that $\tau = 0.01\,\text{s}$, half of the first one. This time we assume the solution $v_C(t) = Ke^{-(t-0.1)/\tau} + A$. Notice that we could also have gone without the time shift in the exponential, as the constant in front would just change to match conditions.

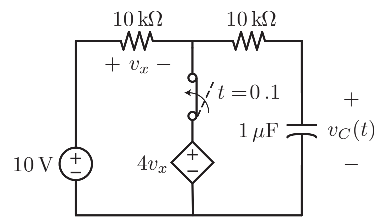

Figure 6: Circuit after $t = 0.1\,\text{s}$

The initial condition is now given by $v_C(0.1)$ from the first solution. This is way past three time constants, so we approximate that $v_C(0.1) \approx 10$. The final condition has changed from $v_C(\infty) = 10$ to $v_C(\infty) = 8$. Using these conditions to solve for the constants gives us $v_C(t) = 2e^{-100(t-0.1)} + 8\,\text{V}$. Thus, the complete solution is

$$v_C(t) = \begin{cases} 10(1 - e^{-50t})\,\text{V} & \text{for } 0 \leq t \leq 0.1\,\text{s} \\ 2e^{-100(t-0.1)} + 8\,\text{V} & \text{for } t \geq 0.1\,\text{s} \end{cases} \tag{23}$$

Notice that while this function is not smooth, it is indeed continuous, as per the restriction that voltage cannot change instantaneously across a capacitor. The behavior is such that it first charges up to 10 V, essentially reaching DC steady-state. Then after the second switch is flipped, we solve a brand new RC circuit problem, settling to the final value of 8 V.

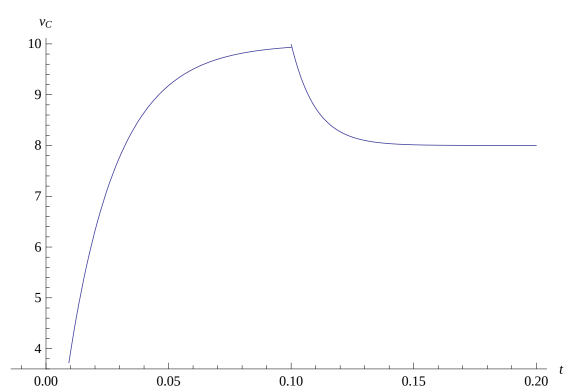

Figure 7: The complete response $v_C(t)$