Thévenin and Norton Equivalent Circuits

Suppose you were given a circuit with two resistors in series, painfully obvious with no tricks or traps. Assuming you didn't care about the node between them, what is the first thing you would do? You'd probably combine them into one resistor by adding the two values together.

Have you ever stopped to think about why you did it or where the rule comes from? Sometimes we are so used to such simple concepts that we forget our motivations for doing so or their physical meaning. When we combine resistors in series and parallel, we do so because we don't necessarily care about any of the nodes and branches inside the *black box* containing the resistive network. We only care about the two terminals that we keep fixed. The resistive network across these terminals is replaced with a single, *equivalent* resistor.

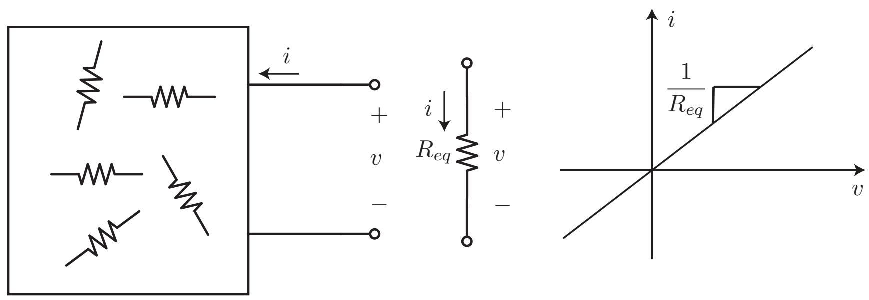

Figure 1: A "black box" of resistors, its equivalent resistance, and their i-v characteristic

But what do we *really* mean by "equivalent"? Well, recall that the series and parallel rules were derived using KCL and KVL. The heart of circuit equivalence lies in the idea of the *i-v relationship*, the two quantities that we can measure from the two terminals of interest. Suppose we applied a voltage v across the original resistive network and observed a current i coming out of the source. Then applying the same v across our one equivalent resistor should give us the same output i. If this holds for any arbitrary v, then we say the two circuits are *equivalent*. (Alternatively, we can probe the circuit with a current input and observe the voltage output.)

Okay, so now we have an idea of where we're going. The Thévenin and Norton circuits can simply be thought of as generalizations of the above idea. Suppose we now have an arbitrary circuit, containing not only resistors but a variety of sources as well. Can we find an "equivalent," simpler circuit to represent it? And our criterion for equivalence is the same as before; the i-v relationship must not change.

That's where Thévenin and Norton come in. Their theorems tell us that the answer is yes, as long as the circuit of interest is *linear*. Any circuit of this type can be redrawn either as a voltage source in series with a resistor, or a current source in parallel with the same resistor. Unlike with just resistors, we now need *two* elements to completely model the original circuit. (Caveat: The polarities of the sources do matter. I_N always points in the direction of increasing V_{Th}.)

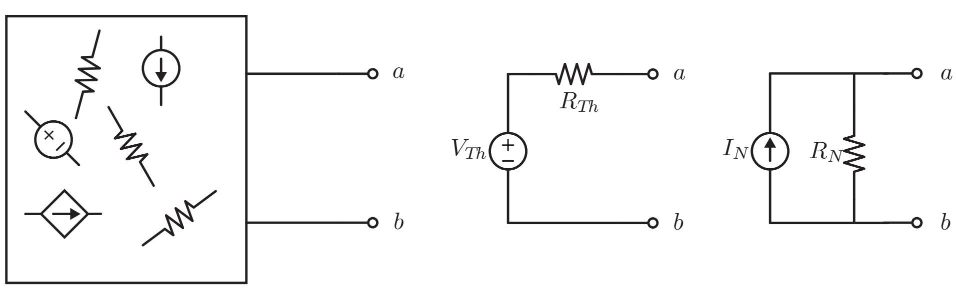

Figure 2: The Thévenin and Norton equivalent circuits

As the circuits are drawn this way, it turns out that finding the values of these circuits is relatively simple. Notice that the Thévenin voltage V_{Th} is just the open-circuit voltage v_{oc} across the terminals, while $v_{oc} = I_N R_N$ for the Norton circuit. In addition, if we short the two terminals of our circuits, we get that the short-circuit current i_{sc} across the Thévenin circuit is V_{th}/R_{Th}, while for the Norton circuit $I_N = i_{sc}$. Not only do these relationships tell us how to find these quantities, it also tells us that the Thévenin and Norton circuits are tightly coupled by the formulae $V_{Th} = I_N R_{Th}$ and $R_{Th} = R_N$.

At this point, it is worthwhile to take a step back to think about why we opted to choose v_{oc} and i_{sc} as our default go-to values. Recall that we desire the i-v relationship to hold for any arbitrary v and i. This *includes* the cases of v_{oc}, in which $i = 0$, and i_{sc}, in which $v = 0$. What we've effectively done is take only two specific instances and solve for a function that supposedly works for *all* instances of voltage (or current) inputs. How could we have done this?

Remember the little specification that the circuits we're looking at must be linear? This is where it comes in! Because the circuits are linear in one input, the equivalent circuit (or the mathematical function describing it) is *completely defined* with only two instances. Consider actually attaching an arbitrary voltage v to the Thévenin and Norton circuits as shown.

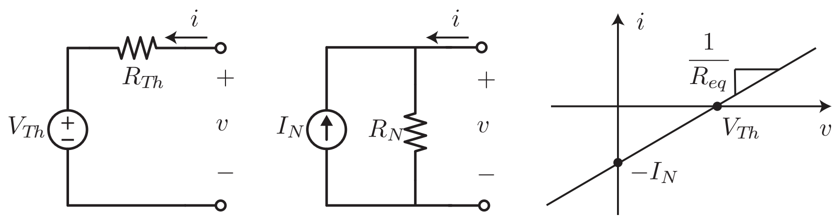

Figure 3: The i-v characteristic of the equivalent circuits

We can find an analytical expression relating i and v for both circuits, in terms of the Thévenin and Norton quantities. For the Thévenin, Ohm's law gives us

$$i = \frac{v - V_{Th}}{R_{Th}} = \frac{v}{R_{Th}} - I_N \tag{1}$$

For the Norton case, we can apply KCL on the top node, coupled with Ohm's law across R_N:

$$i = \frac{v}{R_N} - I_N = \frac{v}{R_{Th}} - I_N \tag{2}$$

As expected, both circuits give us the same relationship. Moreover, they are clearly linear in i and v. The quantities that we normally find, $V_{Th} = v_{oc}$ and $I_N = i_{sc}$, lie on the axes, while the inverse of R_{Th} forms the slope. Together with the circuits themselves, these i-v plots are equivalently powerful ways of representing the circuit. Indeed, in this form it is very easy to see that the whole concept is just a generalization of equivalent resistances in Figure 1. In the resistance case, it was just that $V_{Th} = I_N = 0$.

These relationships also give us an alternative definition of R_{Th}. Using our equations, we can see that $R_{Th} = \frac{v}{i}$ when $V_{Th} = I_N = 0$. This stipulation means that all independent sources are zeroed out in the original circuit. Again, since all of these equations must hold for any arbitrary v and i, this gives us a way to find R_{Th} directly:

1. Zero out all independent sources in the circuit.
2. Attach some arbitrary v_{test} (or i_{test}) to the terminals of interest with the polarity shown in Figure 3 above. This may be left symbolic, or it may be any numeric value.
3. Find the response i_{test} (or v_{test}).
4. The Thévenin resistance R_{Th} is equal to the ratio $\frac{v_{test}}{i_{test}}$.

In the absence of any dependent sources, the resulting circuit after step 1 will contain only resistors. Thus, steps 2-4 in this case can be reduced to simply finding R_{eq} between the two terminals, since this is the very definition of R_{eq} that we had established at the beginning of this discussion!